AT–398

B.Sc. (Part-III) Semester-VI Examination

MATHEMATICS

Linear Algebra

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt this question once only.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative :

(1) Any superset of a linearly dependent set is : 1

(a) Linearly independent

(b) Linearly dependent

(c) Linearly independent and linearly dependent

(d) None of these

(2) If U and W are the subspaces of a vector space V(F) then $U \cup W$ is a subspace iff : 1

(a) $U \subseteq W$ or $W \subseteq U$ (b) $U \supseteq W$ or $W \supseteq U$

(c) $U \cap W = \{0\}$ (d) None of these

(3) If $T : U \to V$ be a linear map then R(T) is a subspace of : 1

(a) U (b) V

(c) $U \cap V$ (d) None of these

(4) If U, V be finite dimensional vector space and $T : U \to V$ be a linear one-one and onto map then : 1

(a) dim U = dim V (b) U = V

(c) dim U ≠ dim V (d) U ≠ V

(5) An element of dual space of V is called a : 1

(a) Linear element (b) Bilinear element

(c) Linear functional (d) None of these

(6) Eigen vectors corresponding to distinct eigen values of a square matrix are : 1

(a) Linearly independent

(b) Linearly dependent

(c) Linearly independent as well as linearly dependent

(d) None of these

(7) In an inner product space V, the inequality $|(u, v)| \le \|u\| \cdot \|v\|$, for all $u, v \in V$ is known as : 1

(a) Triangular inequality

(b) Cauchy-Schwartz inequality

(c) Bessel's inequality

(d) None of these

(8) If W is a subspace of an inner product space V and $W^{\perp}$ is orthogonal complement of W, then : 1

(a) $W^{\perp}$ is a subspace of W

(b) $W \cap W^{\perp} = \{0\}$

(c) $W \cap W^{\perp} \neq \{0\}$

(d) None of these

(9) If A is any submodule of a R-module M, then the zero element of the quotient group M/A is : 1

(a) M

(b) A

(c) $\{0\}$

(d) None of these

(10) Let $T : M \to H$ be a homomorphism of a R-module M into R-module H, then : 1

(a) R(T) is a subset of M

(b) R(T) is a submodule of M

(c) R(T) is a submodule of H

(d) None of these

UNIT—I

2. (a) Define Linear span. If S be a non-empty subset of a vector space V, then prove that [S] is the smallest subspace of V containing S. 1+3

(b) Prove that an arbitrary intersection of subspaces of a vector space is again a subspace. 3

(c) Prove that the set of functions $\{x, |x|\}$ is L.I. in a real vector space of the continuous functions defined on $(-1, 1)$. 3

3. (p) If U and W are finite dimensional subspaces of a vector space V, then prove that :

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$ 6

(q) Given two LI vectors (1, 0, 1, 0), (0, -1, 1, 0) of V_4. Find a basis of V_4 that includes these two vectors. 4

UNIT—II

4. (a) If T is a linear transformation of V_2 to V_2 defined by T(2, 1) = (3, 4), T(–3, 4) = (0, 5), then express (0, 1) as a LC of (2, 1) and (–3, 4). Hence find image of (0, 1) under T. 3

(b) Let $T : U \to V$ be a linear map. Then prove that N(T) is a subspace of U. 3

(c) Let $T : V_4 \to V_3$ be a linear map defined by :

$$T(e_1) = (1, 1, 1),\ T(e_2) = (1, -1, 1),\ T(e_3) = (1, 0, 0),\ T(e_4) = (1, 0, 1).$$

Verify Rank-Nullity theorem. 4

5. (p) If $T : U \to V$ be a non-singular linear map, then prove that $T^{-1} : V \to U$ is also a non-singular linear map. 3

(q) If the matrix of a linear map T with respect to bases B_1 and B_2 is $\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

where $B_1 = \{(1, 2, 0), (0, -1, 0), (1, -1, 1)\}$ and $B_2 = \{(1, 0), (2, -1)\}$.

Find T(x, y, z). 4

(r) Find the range, kernel, rank and nullity of a matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and verify Rank-Nullity theorem. 3

UNIT—III

6. (a) Let V be the finite dimensional vector space over F. Then prove that $V \approx \hat{\hat{V}}$. 4

(b) If V is finite dimensional and $V_1 \neq V_2$ are in V, prove that there is an $f \in \hat{V}$ such that $f(V_1) \neq f(V_2)$. 3

(c) Prove that A(W) is a subspace of $\hat{V}$. 3

7. (p) Define Annihilator W. If V be a vector space over F for a subset S of V and $A(S) = \{f \in \hat{V} / f(s) = 0,\ \forall\ s \in S\}$, then prove that A(S) = A(L(S)), where L(S) is linear span of S. 1+3

(q) If U, V are finite dimensional complex vector spaces and $A : U \to V$, $B : U \to V$ are linear maps with $\alpha \in C$, then prove that $(A + B)^* = A^* + B^*$. 3

(r) If W_1 and W_2 are subspaces of a finite dimensional vector space V over F then describe $A(W_1 \cap W_2)$ in terms of $A(W_1)$ and $A(W_2)$. 3

UNIT—IV

8. (a) In an IPS V over F, prove the parallelogram law :

$$\| u + v \|^2 + \| u - v \|^2 = 2(\| u \|^2 + \| v \|^2).$$ 5

(b) Apply Gram-Schmidt method to orthonormalise the set :

$\{(1, 0, 1, 1), (-1, 0, -1, 1), (0, -1, 1, 1)\}$. 5

9. (p) Let $W^{\perp}$ be the set of orthogonal vectors in an IPS V, then prove that $W^{\perp}$ is a subspace of V. 3

(q) Let V be a finite dimensional inner product space. Then prove that V has an orthogonal set as a basis. 4

(r) In $F^{(n)}$ define, for $u = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ and $v = (\beta_1, \beta_2, \ldots\ldots, \beta_n)$,

$$(u, v) = \alpha_1\overline{\beta}_1 + \alpha_2\overline{\beta}_2 + \ldots\ldots\ldots + \alpha_n\overline{\beta}_n .$$

Show that this defines an inner product. 3

UNIT—V

10. (a) Let T be a homomorphic of R-module M into an R-module H. Then prove that T is one-one iff Ker T = {0}. 3

(b) If M_1 and M_2 are submodules of R-module M_1, then prove that $M_1 + M_2$ is a submodule of M. Moreover $M_1 + M_2$ is direct sum of M_1 and $M_2 \Leftrightarrow M_1 \cap M_2 = \{0\}$. 4

(c) If T is a homomorphism of an R-module M to an R-module H, then show that :

(i) $T(o) = o$

(ii) $T(-m) = -Tm, \forall m \in M$

(iii) $T(m_1 - m_2) = Tm_1 \quad Tm_2, \forall m_1, m_2 \in M$. 3

11. (p) If A and B are submodules of M, then prove that $\frac{A+B}{B}$ is isomorphic to $\frac{A}{A \cap B}$. 5

(q) Define submodule of a module. Prove that arbitrary intersection of submodules of a module is a submodule. 5