AS–1471

B.Sc. (Part—III) Semester—VI Examination

MATHEMATICS

(Linear Algebra)

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note : (1) Question No. **1** is compulsory and attempt this question once only.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative :

(i) If S is non empty subset of vector sapce V, then L(S) is ………… . 1

(a) Largest subspace of V containing S.

(b) Smallest subspace of V containing S.

(c) Smallest subspace of V containing V.

(d) None of these.

(ii) The basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)} of the vector space $R^3(R)$ is known as ……… 1

(a) Normal basis (b) Standard basis

(c) Quotient basis (d) Hamel basis

(iii) If U, V be finite dimensional vector spaces and $T : U \to V$ be a linear, one-one and onto map, then ………… . 1

(a) dim U = dim V (b) U = V

(c) dim U ≠ dim V (d) U ≠ V

(iv) The kernel of a linear transformation $T : U \to V$ is a subset of ………… . 1

(a) U (b) V

(c) U and V (d) None of these

(v) An element of dual space of V is called a ……… . 1

(a) Linear element (b) Bilinear element

(c) Linear functional (d) None of these

(vi) Eigen vectors corresponding to distinct eigen values of a square matrix are 1

(a) Linearly independent

(b) Linearly dependent

(c) Linearly independent as well as Linearly dependent

(d) None of these.

(vii) If $\| V \| = 1$, then V is called 1

(a) Normalised (b) Orthonormal

(c) Scalar inner product (d) Standard inner product.

(viii) In an inner product space V(F), following relation : 1

$\| u + v \|^2 + \| u - v \|^2 = 2 (\| u \|^2 + \| v \|^2)$ is called ....

(a) Schwrtz's inequality (b) Triangular law

(c) Parallelogram Law (d) Bessel's inequality

(ix) If ring R has a unit element 1 and $1 \,.\, a = a$, for all $a \in M$, then M is called 1

(a) Unital R-module (b) Left R-module

(c) Unique R-module (d) None of these

(x) If M is any R-module, then M and {0} are always submodules of M these are called .... submodules of M : 1

(a) Proper (b) Improper

(c) Subproper (d) Irreducible.

UNIT—I

2. (a) Let R^+ be the set of all positive real number. Define the operations of vector addition $\oplus$ and scalar multiplication $\otimes$ as follows :

$$u \oplus v = uv, \forall u, v \in R^+$$

and $\alpha \otimes u = u^{\alpha}, \forall u \in R^+ \alpha \in R$.

Prove that R^+ is a real vector space. 5

(b) Let U and W be two subspaces of a vector space V and $Z = U + W$. Then prove that $Z = U \oplus W \Leftrightarrow z = u + w$ is unique representation for any $z \in Z$ and for some $u \in U$, $\omega \in W$. 5

3. (p) Prove that the intersection of two subspaces of a vector space is again a subspace. Is this statement true for union ? 5

(q) Show that the ordered set S = {(1, 1, 0), (0, 1, 1), (1, 0, –1), (1, 1, 1)} is LD and locate one of the vectors from S that belongs to the span of the previous ones. Find also the largest LI subset of S whose span is [S]. 5

UNIT—II

4. (a) Find a linear transformation T from V_2 to V_2 s.t. T (1, 0) = (1, 1) and T (0, 1) = (–1, 2). Prove that T maps the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1) into a parrallelogram. 3

(b) Let T : U → V be a linear map. Then prove that R(T) is a subspace of V. 3

(c) Find the range, kernel, rank and nullity of the matrix :

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 5 \end{bmatrix}$$

and verify Rauk-Nullity theorem. 4

5. (p) Find the matrix of the linear map $T : V_2 \to V_3$ defined by T (x, y) = (– x + 2y, y, – 3x + 3y) related to the bases

$$B_1 = \{(1, 2), (-2, 1)\}$$

and $B_2 = \{(-1, 0, 2), (1, 2, 3), (1, -1, 1)\}$. 5

(q) State and prove Rank-Nullity theorem. 5

UNIT—III

6. (a) Let V be a finite dimensional vector space over F, then prove that $V \approx \hat{V}$. 4

(b) If W_1 and W_2 are subspaces of a finite dimensional vector space V over F, then show that

$$A(W_1 \cap W_2) = A(W_1) + A(W_2).$$ 3

(c) Prove that annihitator of W = A(W) is a subspace of $\hat{V}$. 3

7. (p) Let U, V be finite dimensional complex vector spaces and A : U → V, B : U → V be linear maps of α ε C, then prove that :

(i) $(A + B)^* = A^* + B^*$,

(ii) $(\alpha A)^* = \bar{\alpha} A^*$. 3+2

(q) If W is a subspace of a finite dimensional vector space V, then prove that

$A(A(W)) = W$. 5

UNIT—IV

8. (a) Let V be a set of all continuous complex valued functions on the closed interval $[0, 1]$. If $f(t), g(t) \varepsilon V$, defined by

$$(f(t), g(t)) = \int_0^1 f(t) \cdot \bar{g}(t)\,dt, \text{ then ,}$$

show that this defines an inner product on V. 5

(b) Using Gram-Schmidt Orthogenalisation process orthonormalise the L.I. subset $\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ of V_3. 5

9. (p) If $\{x_1, x_2, \ldots, x_n\}$ be an orthogonal set, then prove that :

$\| x_1 + x_2 + \ldots + x_n \|^2 = \| x_1 \|^2 + \| x_2 \|^2 + \ldots + \| x_n \|^2$. 3

(q) Prove that in an inner product space V,

(i) $\| \alpha u \| = | \alpha | \| u \|$,

(ii) $\| u + v \| \leq \| u \| + \| v \|$. 4

(r) If V is a finite dimensional inner product space and W is a subspace of V then show that $(W^{\perp})^{\perp} = W$. 3

UNIT—V

10. (a) Prove that arbitrary intersection of submodules of a module is a submodule. 3

(b) Let M be an R-module. Then prove the following :

(i) $\gamma \cdot 0 = 0 \quad , \forall \gamma \varepsilon R$

(ii) $-(\gamma \cdot a) = \gamma \cdot (-a) = (-\gamma) \cdot a, \ \forall \gamma \varepsilon R$ and $m \varepsilon M$. 4

(c) If A be a submodule of unital R-module M, then prove that M/A is also unital R-module. 3

11. (p) Define R-module homomorphism. If $T : M \rightarrow H$ be an R-module homomorphism, then prove that :

(i) K(T) is a submodule of M and R(T) is submodule of H.

(ii) T is one-one $\Leftrightarrow K(T) = \{0\}$. 1+4

(q) Let M be an R-module. If H and K are submodules of M with $K \subset H$. Then prove that

$\frac{M}{H} \cong \frac{M/K}{H/K}$. 5