

B.Sc. Part-III (Semester-VI) Examination**MATHEMATICS****(Linear Algebra)****Paper—XI**

Time : Three Hours]

[Maximum Marks : 60

Note :— (1) Question No. 1 is compulsory and attempt this question once only.(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative (1 mark each) :

(i) S is a non-empty subset of vector space V , then the smallest subspace of V containing S is :

- (a) S (b) $\{S\}$
 (c) $[S]$ (d) None

(ii) Let U and V be finite dimensional vector spaces and $T : U \rightarrow V$ be a linear map one-one and onto, then :

- (a) $\dim U = \dim V$ (b) $\dim U \neq \dim V$
 (c) $U = V$ (d) $U \neq V$

(iii) Let W is subspace of vector space V . Then $\{f \in \hat{V} / f(w) = 0, \forall w \in W\}$ is called as :

- (a) Hilatory of W (b) Annihilator of W
 (c) Dual space of W (d) None

(iv) The normalized vector $(1, -2, 5)$ is :

- (a) $(1, -2, 5)$ (b) $\left(\frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)$
 (c) $\left(\frac{1}{2}, -1, \frac{5}{2}\right)$ (d) $\left(\frac{1}{5}, \frac{-2}{5}, 1\right)$

(v) In IPS $V(F)$ the relation $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$ is called as :

- (a) Schwartz inequality (b) Triangle law
 (c) Parallelogram law (d) Bessels inequality

(vi) For two subspaces U and W of $V(F)$, $V = U \oplus W \Leftrightarrow \dots\dots\dots$

- (a) $U \cap W = \{0\}$ (b) $V = U + W$
 (c) $U \cap W = \{0\}$ and $V = U + W$ (d) None of these

- (vii) Let $T : M \rightarrow N$ be an R -homomorphism. If B is a submodule of N , then :
- (a) $T^{-1}(B)$ is submodule of N (b) $T^{-1}(B)$ is submodule of M
 (c) $T^{-1}(B)$ is kernel of R -homomorphism (d) $T^{-1}(B) = T(M)$
- (viii) If $T : U \rightarrow V$ then the set $\{T(u) \mid u \in U\} = \dots\dots\dots$
- (a) $\text{Ker } T$ (b) $R(u)$
 (c) $R(T)$ (d) None of these
- (ix) If $\|V\| = 1$, then V is called :
- (a) Normalised (b) Orthonormal
 (c) Scalar inner product (d) Standard inner product
- (x) If \hat{V} is n -dimensional, then the dimension of V is :
- (a) Less than n (b) Greater than n
 (c) Equal n (d) Zero

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UNIT—I

2. (a) Let U and W be two subspaces of a vector space V and $Z = U + W$. Then prove that $Z = U \oplus W$ iff $z \in Z, z = u + w$ is unique representation for $u \in U$ and $w \in W$. 5
- (b) Extend the linearly independent set $\{(1, 1, 1, 1) (1, 2, 1, 2)\}$ in V_4 to a basis for V_4 . 5
3. (p) If U and W are finite dimensional subspaces of vector space V , then prove that :
- $$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$
- 5
- (q) Let R^+ be the set of all positive real numbers. Define the operations of addition \oplus and scalar multiplication \otimes as follows :
- $$u \oplus v = u \cdot v \quad \forall u, v \in R^+$$
- and $\alpha \otimes u = u^\alpha, \quad \forall u \in R^+ \text{ and } \alpha \in R$
- Prove that R^+ is a real vector space. 5

UNIT—II

4. (a) If U, V is a vector space over a field F and $T : U \rightarrow V$ be a linear, then prove that :
- $$T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n)$$
- $\forall u_i \in U, \alpha_i \in F, 1 \leq i \leq n \text{ and } n \in N.$
- 2
- (b) Let $T : V_4 \rightarrow V_3$ be a linear map defined by $T(e_1) = (1, 1, 1), T(e_2) = (1, -1, 1), T(e_3) = (1, 0, 0), T(e_4) = (1, 0, 1).$
- Verify Rank-nullity theorem. 4

- (c) Find the matrix of the linear map $T : V_2 \rightarrow V_3$ defined by $T(x, y) = (-x + 2y, y, -3x + 3y)$ related to the bases $B_1 = \{(1, 2), (-2, 1)\}$ and $B_2 = \{(-1, 0, 2), (1, 2, 3), (1, -1, 1)\}$.

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5. (p) Let U and V be vector spaces over the same field F . Then prove that function $T : U \rightarrow V$ is linear iff $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$, $\forall \alpha, \beta \in F$ and $u, v \in U$.

5

- (q) If matrix of a linear map T with respect to bases B_1 and B_2 is :

$$\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

where $B_1 = \{(1, 2, 0), (0, -1, 0), (1, -1, 1)\}$ and $B_2 = \{(1, 0), (2, -1)\}$, then find $T(x, y, z)$.

5

UNIT—III

6. (a) Let V be the space of all real valued continuous functions of real variable. Define $T : V \rightarrow V$ by

$$(Tf)(x) = \int_0^x f(t) dt, \forall f \in V, x \in \mathbb{R}.$$

Show that T has no eigen value.

5

- (b) Prove that if V be a finite dimensional vector space over F and $v(\neq 0) \in V$, then $\exists f \in \hat{V}$ such that $f(v) \neq 0$.

5

7. (p) If W_1 and W_2 are subspaces of a finite dimensional vector space V , show that $A(W_1 + W_2) = A(W_1) \cap A(W_2)$.

5

- (q) If K_λ is eigenspace, then prove that K_λ is a subspace of vector space V .

3

- (r) Define characteristic root and characteristic vector.

2

UNIT—IV

8. (a) In $F^{(n)}$ define for $u = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$ and $v = (\beta_1, \beta_2, \dots, \beta_n)$

$$(u, v) = \alpha_1 \bar{\beta}_1 + \alpha_2 \bar{\beta}_2 + \dots + \alpha_n \bar{\beta}_n.$$

Show that this defines an inner product.

4

- (b) If $\{x_1, x_2, x_3, \dots, x_n\}$ be an orthogonal set, then prove that :

$$\|x_1 + x_2 + x_3 + \dots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_n\|^2$$

4

- (c) Prove that orthogonal complement i.e. W^\perp is subspace of V .

2

9. (p) If $\{w_1, w_2, \dots, w_m\}$ is an orthonormal set in V , then $\sum_{i=1}^m |(w_i, v)|^2 \leq \|v\|^2$ for any

$v \in V$.

4

- (q) If V is a finite dimensional inner product space and W is a subspace of V then prove that $(W^\perp)^\perp = W$. 4
- (r) (i) Define inner product in vector space. 1
- (ii) Define orthogonal set. 1

UNIT—V

- 10. (a) Let A be a submodule of an R -module M and T is a mapping from M into M/A defined by $T_m = A + m, \forall m \in M$. Then prove that T is an R -homomorphism of M into M/A and $\ker T = A$. 5
- (b) Let T be a homomorphism of an R -module M to an R -module H . Prove that T is one-one iff $\ker T = \{0\}$. 3
- (c) Define :
 - (i) Submodule
 - (ii) Unital R -module. 2
- 11. (p) If A and B are submodules of M , then prove that $\frac{A+B}{B}$ is isomorphic to $\frac{A}{A \cap B}$. 6
- (q) Prove that arbitrary intersection of submodules of a module is a submodule. 4