AV–1810

B.Sc. (Part—III) Semester-VI Examination

MATHEMATICS (NEW)

(Linear Algebra)

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory. Attempt it once only.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative : 10

(i) A non empty subset U of a vector space V(F) is a subspace of V iff :

(a) $\alpha\beta + uv \in U$ (b) $\alpha u + \beta v \in V$

(c) $\alpha u + \beta . v \in U$ (d) $\alpha u - \beta v \in V$ for all $\alpha, \beta \in F$ and $u, v \in U$

(ii) Any subset of linearly independent set is :

(a) linearly dependent

(b) linearly dependent and linearly independent

(c) linearly independent

(d) None of these

(iii) If T : u → v is linear map then R(T) is subset of :

(a) V (b) $U \cap V$

(c) U (d) $U \cup V$

(iv) An element of dual space V is called a :

(a) Linear element (b) Linear functional

(c) Bilinear element (d) None of these

(v) If u, v be finite dimensional vector spaces and T : u → v be a linear one-one and onto map, then :

(a) dim U = dim V (b) dim U ≠ dim V

(c) U = V (d) U ≠ V

(vi) If V is the finite dimensional vector space over F then :

(a) $V \cong \hat{V}$ (b) $V \neq \hat{V}$

(c) $\hat{V} = \{0\}$ (d) None of these

(vii) If $\| V \| = 1$ then V is called :

(a) Orthogonal (b) Null vector

(c) Normalised (d) None of these

(viii) The normalised vector of (1, −2, 5) is :

(a) $\left(\frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)$ (b) $\left(\frac{1}{2}, -1, \frac{5}{2}\right)$

(c) $\left(\frac{1}{5}, \frac{-2}{5}, -1\right)$ (d) (1, −2, 5)

(ix) R-Module homomorphism is linear transformation if :

(a) R-with unit element (b) R is commutative

(c) R-is a field (d) None of these

(x) If the ring R has a unit element 1 and 1.a = a for all $a \in M$ then M is called :

(a) A unital R-module (b) Right R-module

(c) Left-R-module (d) None of these

UNIT—I

2. (a) Define a basis of a vector space. If $\{v_1, v_2, \ldots, v_n\}$ is a basis of V over F and if $w_1, w_2, \ldots, w_m \in V$ are L.I. over F, then prove that $m \leq n$. 1+4

(b) Define a subspace of a vector space and prove that the non empty subset U of a vector space V(F) is a subspace of V iff $\alpha u + \beta v \in U \ \forall \ \alpha, \beta \in F, u, v \in U$. 1+4

3. (p) Prove that the intersection of two subspaces of a vector space is again a subspace. Is the statement true for union ? Justify. 5

(q) Find span of S = {(1, 2, 1), (1, 1, −1), (4, 5, −2)} and then prove that (2, −1, −8) belongs to the span of S^1 but (1, −3, 5) does not belongs to span of S. 5

UNIT—II

4. (a) Let U, V are the vector spaces over a field F and $T : u \to v$ be a linear map. Then prove that :

(i) $T(0) = 0$

(ii) $T(-u) = -T(u) \ \forall \ u \in U$

(iii) $T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots\ldots + \alpha_n T(u_n)$

$\forall \ u_i \in U, \alpha_i \in F, 1 \leq i \leq n$ and $n \in N$.

5

(b) Let $T : V_4 \to V_3$ be a linear map defined by $T(e_1) = (1, 1, 1)$, $T(e_2) = (1, -1, 1)$, $T(e_3) = (1, 0, 0)$, $T(e_4) = (1, 0, 1)$. Verify Rank-Nullity theorem. 5

5. (p) State and prove Rank-Nullity Theorem. 5

(q) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a matrix of linear map T with respect to bases B_1 and B_2 where $B_1 = \{(1, 1, 1), (1, 0, 0), (0, 1, 0)\}$, $B_2 = \{(1, 2, 3), (1, -1, 1), (2, 1, 1)\}$. Find $T : V_3 \to V_3$ such that $A = (T : B_1, B_2)$. 5

UNIT—III

6. (a) Let V be a finite dimensional vector space over F. Then prove that $V \approx \hat{V}$. 5

(b) Define Annihilator of W. Prove that annihilator of W = A(W) is a subspace of $\hat{V}$. 5

7. (p) If U and V are finite dimensional complex vector spaces and $A : U \rightarrow V$, $B : U \rightarrow V$ are linear maps, then prove that (i) $(A + B)^* = A^* + B^*$, (ii) $(\alpha A)^* = \overline{\alpha} A^*$. 5

(q) If S is a subset of a vector space V and $A(S) = \{f \in \hat{V} / f(x) = 0 \ \forall\ s \in S\}$ then prove that A(S) = A(L(S)) where L(S) is the linear span of S. 5

UNIT—IV

8. (a) State and prove Cauchy-Schwarz inequality. 5

(b) (i) If $\{x_1, x_2, \ldots\ldots, x_n\}$ is an orthogonal set, then prove that :

$$\| x_1 + x_2 + x_3 + \ldots\ldots + x_n \|^2 = \| x_1 \|^2 + \| x_2 \|^2 + \ldots\ldots\ldots + \| x_n \|^2.$$

(ii) Prove that every orthogonal set is LI. 5

9. (p) Let V be a finite dimensional inner product space. Then prove that V has an orthogonal (orthonormal) set as a basis. 5

(q) Using Gram-Schmidt process, orthonormalise the set of vectors :

$\{(1, 0, 1, 0), (1, 1, 3, 0), (0, 2, 0, 1)\}$ of V_4. 5

UNIT—V

10. (a) If M_1 and M_2 are submodules of R-module M, then prove that $M_1 + M_2$ is a sub module of M. Moreover prove that $M_1 + M_2$ is a direct sum of M_1 and M_2 iff $M_1 \cap M_2 = \{0\}$. 5

(b) Define :

(i) R-module homomorphism

(ii) Quotient module

and prove that if A be a submodule of unital R module M, then prove that M/A is also unital R-module. 1+1+3

11. (p) If H and K are submodules of M then prove that $\dfrac{H+K}{K} \cong \dfrac{H}{H \cap K}$. 5

(q) If T is a homomorphism of a R-module M to R-module H then prove that :

(i) $T(0) = 0$

(ii) $T(-m) = -T(m) \ \forall\ m \in M$

(iii) $T(m_1 - m_2) = T(m_1) - T(m_2) \ \forall\ m_1 m_2 \in M$. 3

(r) If M be an R-module and $m \in M$. Then prove that $A = \{rm / r \in R)$ is a submodule of M. 2