AU–153

B.Sc. (Part-III) Semester-VI Examination

MATHEMATICS (Old) (Upto Winter 2018)

(Linear Algebra)

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note :—(1) Question No. 1 is compulsory and attempt this question once only.

(2) Attempt **ONE** question from each unit.

1. (i) Let U and W be two distinct subspaces of an n-dimensional vector space V and dim U = dim W = n – 1. Then the dim $(U \cap W)$ is : 1

(a) n – 2 (b) n

(c) n – 4 (d) n – 3

(ii) The basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)} of the vector space $R^3(R)$ is known as : 1

(a) Quotient basis (b) Normal basis

(c) Standard basis (d) None of these

(iii) $\{T(u) \mid u \in U\}$ = ... 1

(a) Ker(T) (b) R(T)

(c) R(u) (d) None of these

(iv) If U, V be finite dimensional vector space and $T : U \to V$ be a linear, one-one and onto map then : 1

(a) dim U = dim V (b) dim U ≠ dim V

(c) U = V (d) U ≠ V

(v) Let U and V be complex vector spaces. If $A : U \to V$ be a linear map, then adjoint of A i.e. A^* is a linear map : 1

(a) From $\hat{U}$ to $\hat{V}$ (b) From $\hat{V}$ to $\hat{U}$

(c) From U to V (d) From V to U

(vi) If U and W are subspaces of V over F then $U \subseteq W \Rightarrow$ 1

(a) $A(U) = A(W)$ (b) $A(W) \subseteq A(U)$

(c) $A(U) \simeq A(W)$ (d) $A(U) \supseteq A(W)$

(vii) An element of dual space of V is called a : 1

(a) Linear functional (b) Linear element

(c) Bilinear element (d) None of these

(viii) If W is a subspace of an inner product space V and $W^\perp$ is orthogonal complement of W then : 1

(a) $W \cap W^\perp \neq \{0\}$ (b) $W \cap W^\perp = \{0\}$

(c) $W^\perp$ is a subset of W (d) None of these

(ix) If ring R has a unit element 1 and 1.a = a, for all $a \in M$, then R-module M is called : 1

(a) Unique R-module (b) Unital R-module

(c) Left R-module (d) None of these

(x) Let $T : M \to H$ be a homomorphism of a R-module M into R-module H, then : 1

(a) R(T) is a submodule of M (b) R(T) is a submodule of H

(c) R(T) is a subset of M (d) None of these

UNIT—I

2. (a) If V is a vector space over F, then prove that :

(i) $\alpha\, 0 = 0 \ \forall\ \alpha \in F$

(ii) $0\, v = 0 \ \forall\ v \in V$

(iii) $(-\alpha)\, v = -(\alpha v) \ \forall\ \alpha \in F, \forall\ v \in V$

(iv) $\alpha\, v = 0 \Leftrightarrow \alpha = 0$ or $v = 0$, $(\alpha \in F, v \in V)$ 5

(b) Show that a non empty subset U of a vector space V over F is a subspace of V iff :

(i) $u + v \in U \ \forall\ u, v \in U$ and

(ii) $\alpha\, u \in U \ \forall\ \alpha \in F, u \in U$. 5

3. (p) If U and W be subspaces of a vector space V_3, where :

$U = \{(x_1, x_2, x_3) \in V_3 \mid x_3 = x_1 + x_2\}$

$W = \{(x_1, x_2, x_3) \in V_3 \mid x_1 = x_2 = x_3\}$ then

show that V_3 is the direct sum of U and W. 5

(q) Show that the ordered set :

$S = \{(1, 1, 2), (1, -1, 1), (1, 3, 3), (-1, 3, 0)\}$ is LD and locate one of the vectors that belongs to the span of previous one. Find also the largest LI subset whose span is equal to [S]. 5

UNIT—II

4. (a) If T is a linear transformation from V_2 to V_2 defined by $T(2, 1) = (3, 4)$, $T(-3, 4) = (0, 5)$, then express (0, 1) as a LC of (2, 1) and (–3, 4).

Hence find image of (0, 1) under T. 5

(b) Let $T : U \to V$ be a linear map.

If T is 1 – 1 and $u_1, u_2, \ldots\ldots, u_n$ are LI vectors in U, then prove that $Tu_1, Tu_2, \ldots\ldots, Tu_n$ are LI vectors in V. 5

5. (p) Find the range, kernel and nullity for the linear map $T : V_3 \to V_3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3 - 2x_1)$. Also verify Rank-nullity theorem. 5

(q) If matrix of a linear map T with respect to bases B_1 and B_2 is $\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$,

where :

$B_1 = \{(1, 2, 0), (0, -1, 0), (1, -1, 1)\}$ and

$B_2 = \{(1, 0), (2, -1)\}$. Find T(x, y, z). 5

UNIT—III

6. (a) Let V be a vector space over F. For a subset S of V, let $A(S) = \{ f \in \hat{V} \mid f(s) = 0 \ \forall\ s \in S$.

Prove that A(S) = A(L(S)), where L(S) is linear span of S. 5

(b) Define Annihilator of W = A(W). 1

Prove that annihilator of W = A(W) is a subspace of $\hat{V}$. 4

7. (p) If W_1 and W_2 are subspaces of a finite dimensional vector space V, describe $A(W_1 + W_2)$ in terms of $A(W_1)$ and $A(W_2)$. 5

(q) If S is a subset of a vector space V and $A(S) = \{ f \in \hat{V} \mid f(x) = 0 \ \forall \ s \in S$, then prove that $A(S) = A(L(S))$, where L(S) is the linear span of S. 5

UNIT—IV

8. (a) Prove that $W^{\perp}$ is a subspace of V. 5

(b) Let V be an inner product space over F. In V define the distance d(u, v) from u to v by $d(u, v) = \| u - v \|$. Prove that :

(i) $d(u, v) \geq 0$ and $d(u, v) = 0 \Leftrightarrow u = v$

(ii) $d(u, v) = d(v, u)$

(iii) $d(u, v) \leq d(u, w) + d(w, v) \ \forall \ u, v, w \in V$ 5

9. (p) Find the orthonormal basis of $P_2 [-1, 1]$ starting from the basis $\{1, x, x^2\}$ using the inner product defined by $(f, g) = \int_{-1}^{1} f(x) g(x) dx$. 5

(q) If $\{x_1, x_2, ..., x_n\}$ be an orthogonal set, then prove that :

$$\| x_1 + x_2 + ... + x_n \|^2 = \| x_1 \|^2 + \| x_2 \|^2 + ... + \| x_n \|^2$$ 5

UNIT—V

10. (a) If R be a ring and $T : M \to H$ be an R-module homomorphism, then prove that $\frac{M}{\text{Ker } T} \cong R(T)$. 5

(b) Prove that every abelian group G is a module over the ring of integers Z. 5

11. (p) If A is a submodule of an R-module M and T is a mapping from M into M/A defined by $T_m = A + m \ \forall \ m \in M$ then prove that T is an R-homomorphism of M into M/A and Ker T = A. 4

(q) If M is an R-module and $m \in M$ then prove that $\{rm \mid r \in R\}$ is a submodule of M. 3

(r) If λ is a left ideal of R and if M is an R-module, show that for $m \in M$,

$\lambda_m = \{xm \mid x \in \lambda\}$ is a submodule of M. 3