B.Sc. (Part—III) Semester—V Examination
MATHEMATICS (New)
(Mathematical Analysis)
Paper—IX

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Question No. **1** is compulsory. Attempt once.

(2) Attempt **ONE** question from each Unit.

1. Choose the correct alternatives :

(i) Let f: [0, 1] → R be Riemann integrable. Which of the following is always true :

(a) f is continuous

(b) f is monotone

(c) f has only finite number of discontinuities

(d) the set of discontinuties of f may be infinite ? 1

(ii) An improper integral $\int_a^\infty \frac{dx}{x^p}$, $a \in R$ is convergent if :

(a) $p < 1$ (b) $p > 1$

(c) $p \geq 1$ (d) $p = 1$ 1

(iii) $\beta(m, n)$ is :

(a) $\Gamma m \, \Gamma n$ (b) $\frac{\Gamma(m+n)}{\Gamma m \, \Gamma n}$

(c) $\frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$ (d) $\frac{\Gamma m \, \Gamma n}{\Gamma(m-n)}$ 1

(iv) In the real line R, which of the following is true ?

(a) Every bounded sequence converges (b) Every sequence converges

(c) Every Cauchy sequence converges (d) None of the above 1

(v) Every neighbourhood is a/an :

(a) Closed set (b) Open set

(c) Open closed set (d) None of the above 1

(vi) A function u(x, y) is harmonic in region D if :

(a) $u_{xx} - u_{yy} = 0$ (b) $u_{xy} + u_{yx} = 0$

(c) $u_{xy} - u_{yx} = 0$ (d) $u_{xx} + u_{yy} = 0$ 1

(vii) The function $f(z) = \sqrt{|xy|}$ is ______ at the origin.

(a) Harmonic function (b) Analytic function

(c) Conjugate function (d) Not analytic function 1

(viii) If f(z) and $\overline{f(z)}$ are both analytic functions then f(z) is :

(a) Identically zero (b) Constant

(c) Unbounded (d) None of the above 1

(ix) The points z where $|e^{z^2}| = 10$ form a :

(a) Circle (b) Straight line

(c) Hyperbola (d) Parabola 1

(x) A bilinear transformation with two non-infinite fixed points α and β having Normal form $\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right)$ is Elliptic if :

(a) $|k| \neq 1$, k is real (b) $k \neq 1$, k is not real

(c) $|k| = 1$ (d) None of the above 1

UNIT—I

2. (a) Prove that every continuous function is integrable. 4

(b) Let the function f be defined as :

$$f(x) = 1, \quad \text{when } x \text{ is rational}$$
$$= -1, \quad \text{when } x \text{ is irrational}$$

Show that f is not R-integrable over [0, 1] but $|f| \in R\,[0, 1]$. 3

(c) Show that any constant function defined on a bounded closed interval is integrable. 3

3. (p) If f is a bounded and integrable function over [a, b] and M, m are bounds of f over [a, b], prove that :

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$ 4

(q) Prove that $\frac{2}{17} < \int_{-1}^{2} \frac{x}{1+x^4}\,dx < 1/2$. 3

(r) If f is continuous and non-negative on [a, b], then show that $\int_a^b f(x)\,dx \ge 0$. 3

UNIT—II

4. (a) Prove that the integral $\int_{a^-}^{b} \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geq 1$. 4

(b) Show that $\int_{1}^{\infty} \frac{\sin x}{x^2} dx$ converges absolutely. 3

(c) Show that $\int_{0}^{\infty} e^{-x^2} dx$ converges. 3

5. (p) Prove that $\beta(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$. 4

(q) Prove that $\int_{0}^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta = \frac{\pi}{32}$. 3

(r) Prove that $\Gamma(n+1) = n\Gamma(n)$. 3

UNIT—III

6. (a) If $f(z) = u(x, y) + iv(x, y)$ be analytic in a region D, then prove that $u(x, y)$ and $v(x, y)$ satisfy Cauchy-Riemann equations. 4

(b) If $f(z)$ and $f(\bar{z})$ are analytic functions, prove that $f(z)$ is constant. 3

(c) Show that $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate function. Hence find $f(z) = u + iv$. 3

7. (p) If u and v are harmonic in region R, prove that $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$ is analytic in R. 4

(q) If the function $f(z) = u + iv$ be analytic in domain D then prove that, the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system, where c_1 and c_2 are arbitrary constants. 3

(r) Determine a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic. 3

UNIT—IV

8. (a) Prove that, every bilinear transformation with two non infinite fixed points α, β is of the form $\frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}$, when k is constant. 5

(b) Under the transformation $w = \sqrt{2}e^{i\pi/4}z$, find the image of the rectangle bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 3$. 5

9. (p) Prove that the cross ratio remains invariant under a bilinear transformation. 5

(q) Prove that under the transformation $w = \frac{z - i}{iz - 1}$ the region $I_m(z) \geq 0$ is mapped into the region $|w| \leq 1$. 5

UNIT—V

10. (a) Show that $d(x, y) = |x - y|, \forall x, y \in R$ defines a metric on $\mathbb{R}$. 5

(b) Define :

(i) Limit point

(ii) Boundary point. 2

(c) Prove that every neighbourhood is an open set. 3

11. (p) Define :

(i) Complete metric space

(ii) Open set. 2

(q) Prove that every convergent sequence in a metric space is a Cauchy sequence. 3

(r) Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$ then, prove that $d(x_n, y_n) \to d(x, y)$. 5