AS–1424

B.Sc. Part—II (Semester—IV) Examination

MATHEMATICS (New)

(Modern Algebra : Groups and Rings)

Paper—VII

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt at once only.

(2) Solve **one** question from each unit.

1. Choose the correct alternatives (**1** mark each) : 10

(i) The subgroup N of G is a normal subgroup of G iff :

(a) $gN \neq Ng$ for some $g \in G$ (b) $gN = Ng$ for all $g \in G$

(c) $Ng = N$ for some $g \in G$ (d) $gN = N$ for all $g \in G$

(ii) If H is a subgroup of a group G such that $H \neq \{e\}$ and $H \neq G$ then H is called :

(a) A trivial subgroup (b) Proper subgroup

(c) Improper subgroup (d) None of these

(iii) The product of two odd permutations is :

(a) Odd (b) Even

(c) Both odd and even (d) None of these

(iv) The identity element of the quotient group $G \mid H$ is :

(a) G (b) H

(c) $G \mid H$ (d) $H \mid G$

(v) A homorphism of a group into itself is :

(a) A homomorphism (b) An isomorphism

(c) An endomorphism (d) None of these

(vi) Which of the following is not an integral domain ?

(a) $(C, +, \cdot)$ (b) $(Q, +, \cdot)$

(c) $(R, +, \cdot)$ (d) $(N, +, \cdot)$

(vii) If in a ring R, $x^2 = x \ \forall x \in R$, then R is :

(a) Commutative ring (b) Division ring

(c) Boolean ring (d) Ring with unity

(viii) A field which contains no proper subfield is called :

(a) Subfield (b) Prime field

(c) Integral domain (d) Division ring

(ix) The characteristic of a finite integral domain is :

(a) Even number (b) Odd number

(c) Prime number (d) None of these

(x) A ring which has only trivial ideal is called :

(a) A subring (b) A proper ring

(c) A simple ring (d) None of these

UNIT—I

2. (a) Let G be a group then prove that $(ab)^{-1} = b^{-1} a^{-1} \ \forall a, b \in G$. 3

(b) Prove that every subgroup of a cyclic group is cyclic. 3

(c) Define even and odd permutation. Explain whether the following permutation is even or odd $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 3 & 6 & 1 & 7 & 9 & 8 \end{pmatrix}$. 4

3. (p) Prove that the intersection of any two subgroups of a group G is a subgroup of G. 3

(q) Prove that every permutation on a finite set is either a cycle or it can be expressed as a product of disjoint cycles. 4

(r) Let $G = \{a + b\sqrt{2} \mid a, b \in Q\}$. Show that G is a group w.r. to addition. 3

UNIT—II

4. (a) Let H be a subgroup of a group G and let a, b, $\in$ G. Then prove that Ha = Hb iff $ab^{-1} \in H$. 4

(b) Show that every subgroup of an abelian group is normal. 3

(c) If G = {1, –1, i, –i} and N = {1, –1}, then show that N is a normal subgroup of the multiplicative group G. Also find the quotient group G|N. 3

5. (p) If G is a finite group and H is a subgroup of G, then prove that O(H) is a divisor of O(G). 4

(q) Prove that the subgroup N of G is a normal subgroup of G if and only if each left coset of N in G is a right coset of N in G. 4

(r) Show that if G is abelian, then quotient group G|N is also abelian. 2

UNIT—III

6. (a) Prove that any infinite cyclic group is isomorphic to the additive group of integers. 4

(b) If ϕ is an homomorphism of a group G into a group G', then prove that :

(i) $\phi(e) = e'$

(ii) $\phi(x^{-1}) = (\phi(x))^{-1}\ \forall\ x \in G$ where e and e' are the unit elements of G and G' respectively. 4

(c) Define :

(i) Endomorphism

(ii) Isomorphism. 2

7. (p) If ϕ be a homomorphism of G on to G' with Kernel K, then prove that $G \mid K \approx G'$. 5

(q) Let G is a group of nonzero real numbers under multiplication $\phi : G \rightarrow G$ such that $\phi(x) = x^2\ \forall\ x \in G$, then prove that ϕ is homomorphism and also find its Kernel. 3+2

UNIT—IV

8. (a) If R is a ring in which $x^2 = x \ \forall \ x \in R$, then prove that R is a commutative ring of characteristic 2. 5

(b) Let the integer $n \geq 2$ and $Z_n = \{0, 1, 2, -----, n-1\}$. Show that Z_n is a commutative ring with unity under the addition and multiplication mod n. 5

9. (p) Prove that every prime field of characteristic zero is isomorphic to the field Q of rational numbers. 5

(q) Prove that a finite integral domain is a field. 5

UNIT—V

10. (a) Let R be a ring $a \in R$ and $r(a) = \{x \in R \mid ax = 0\}$. Then prove that r(a) is a right ideal of R. 3

(b) If R is a commutative ring with unity, then prove that every maximal ideal of R is a prime ideal. 3

(c) If U is an ideal of the ring R, then prove that R/U is a ring. 4

11. (p) Let R be a ring. Then prove that the intersection of two left ideals of R is a left ideal of R. 3

(q) Prove that a homomorphism f of a ring R to a ring R' is an isomorphism iff Ker $f = \{0\}$. 4

(r) Prove that the ring of 2×2 matrices of rationals has no ideal other than {0} and the ring itself. 3