AT–348

B.Sc. (Part—II) Semester—IV Examination

MATHEMATICS (NEW)

(Modern Algebra : Groups and Rings)

Paper—VII

Time : Three Hours] [Maximum Marks : 60

Note :—(1) Question No. **1** is compulsory and attempt it once only.

(2) Solve **ONE** question from each unit.

1. Choose the correct alternative (**1** mark each) :

(i) The identity permutation is :

(a) Even (b) Odd

(c) Even and odd (d) None of these

(ii) If N is a normal subgroup of a finite group G, then O(G/N) is equal to :

(a) $O(G) \cdot O(N)$ (b) $O(N) \mid O(G)$

(c) $O(G) \mid O(N)$ (d) None of these

(iii) The product of disjoint cycles is :

(a) Cyclic (b) Not commutative

(c) Commutative (d) None of these

(iv) Let G be a group and let $a \in G$, if $O(a) = 3$ then $O(a^{-1})$ is equal to :

(a) 0 (b) 1

(c) 2 (d) 3

(v) A homomorphism of a group into itself is :

(a) a homomorphism (b) an isomorphism

(c) an endomorphism (d) None of these

(vi) In ring R, $x^2 = x \ \forall \ x \in R$ then R is :

(a) Division ring (b) Boolean ring

(c) Ring with unity (d) Commutative ring

(vii) The ring M of 2 × 2 matrices is :

(a) an integral domain (b) not an integral domain

(c) commutative ring (d) None of these

(viii) An integral domain is :

(a) always a field (b) never a field

(c) a field when it is finite (d) None of these

(ix) A ring which has only trivial ideal is called :

(a) a subring (b) a proper ring

(c) a simple ring (d) None of these

(x) The intersection of two right ideals of a ring R is :

(a) a left ideal of R (b) a right ideal of R

(c) both left and right ideal of R (d) None of these 10

UNIT —I

2. (a) Prove that a group G is abelian iff $(ab)^2 = a^2b^2$, $\forall$ a, b $\in$ G. 3

(b) If S = {1, 2, 3, 4, 5} and f, g be permutations on s given by :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}$$

then prove that the product of permutations is not commutative. 4

(c) Prove that any cyclic group is abelian. 3

3. (p) Show that, a non-empty subset H of a group G is a subgroup of G iff :

(i) a, b $\in$ H $\Rightarrow$ ab $\in$ H,

(ii) a $\in$ H $\Rightarrow$ a^{-1} $\in$ H. 4

(q) If H_1 and H_2 are the subgroups of group G then prove that $H_1 \cap H_2$ is also a subgroup of G. 3

(r) Prove that the product of an even permutation and an odd permutation is odd. 3

UNIT—II

4. (a) Show that if G is abelian, then the quotient group G/N is also abelian. Is its converse true ? Explain. 5

(b) If H is a subgroup of G and N is a normal subgroup of G, then show that $H \cap N$ is a Normal subgroup of H. 5

5. (p) If G is a finite group and H is a subgroup of G then prove that O(H) is a divisor of O(G). 4

(q) Let H be a subgroup of G. If $N(H) = \{g \in G \mid gHg^{-1} = H\}$. Show that N(H) is a subgroup of G. 3

(r) Prove that N is a normal subgroup of G if and only if $gNg^{-1} = N \; \forall \; g \in G$. 3

UNIT—III

6. (a) Show that any infinite cyclic group is isomorphic to the additive group of integers. 4

(b) Let G be any group and g a fixed element in G. Define $\phi : G \to G$ by $\phi(x) = gxg^{-1}$. Prove that ϕ is an isomorphism of G onto G. 4

(c) Let G be a group of non-zero real numbers under multiplication and $\phi : G \to G$ such that $\phi(x) = 2^x \; \forall \; x \in G$ then prove that ϕ is not a homomorphism. 2

7. (p) If M, N are normal subgroups of G, then prove that $\frac{NM}{M} \approx \frac{N}{N \cap M}$. 5

(q) Show that the mapping $f : C \to R$ defined by $f(x + iy) = x$ is a homomorphism of the additive group of complex numbers onto the additive group of real numbers and find the Kernel of f. 5

UNIT—IV

8. (a) Prove that a ring R is commutative iff $(a + b)^2 = a^2 + 2ab + b^2$. 4

(b) Show that intersection of two subrings of a ring is a subring. 3

(c) Let the characteristic of the ring R be 2 and let $ab = ba \; \forall \; a, b \in R$. Then show that $(a + b)^2 = a^2 + b^2$. 3

9. (p) Prove that every prime field of finite characteristics $p > 0$ is isomorphic to the field z_p. 4

(q) If R is a ring with zero element 0, then for all a, b, c $\in$ R. Prove that :

(i) $a \cdot 0 = 0 \cdot a = 0$

(ii) $(-a) \cdot (-b) = a \cdot b$ 4

(r) Prove that a field is an integral domain. 2

UNIT—V

10. (a) If U and V are ideals of a ring R then prove that $U \cap V$ is the largest ideal that is contained in both U and V. 5

(b) In a principle ideal domain if p is prime and $p \mid ab$ then prove that $p \mid a$ or $p \mid b$. 5

11. (p) If U is an ideal of the ring R, then prove that R/U is a ring. 5

(q) If F is a field, then prove that its only ideals are {0} and F itself. 3

(r) Define Maximal ideal. 2