AS–1449

B.Sc. (Part—III) Semester—V Examination

MATHEMATICS

Paper—X

(Modern Algebra)

Time : Three Hours] [Maximum Marks : 60

Note :— Question **ONE** is compulsory and attempt it once and solve **ONE** question from each unit.

1. Choose the correct alternative (1 mark each) :— 10

(1) If N is a normal subgroup of group G, then the factor group G/N is a group with respect to which of the following binary operation.

(a) Na + Nb = Nab (b) Na · Nb = Nab

(c) Na | Nb = Nab (d) Na – Nb = Nab

(2) Group G is abelian group if for all a, b $\in$ G :

(a) $ab^{-1} = ab$ (b) $a^{-1}b = ab^{-1}$

(c) ab = ab (d) ab = ba

(3) Let (G, +) be a group. Then mapping $\phi : G \to G$ is homomorphism if :

(a) $\phi(a + b) = \phi(a) + \phi(b)$ (b) $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$

(c) $\phi(a - b) = \phi(a) - \phi(b)$ (d) $\phi\left(\frac{a}{b}\right) = \phi(a) / \phi(b)$

(4) If $\phi : G \to G'$ is a homomorphism, then ker ϕ is a :

(a) Subgroup of G' (b) Normal subgroup of G

(c) Proper subgroup of G' (d) Quotient subgroup of G

(5) Let G be a group, a $\in$ G be a fixed element and f : G $\to$ G be a mapping defined by $f(x) = axa^{-1}$. Which of the following is not true ?

(a) f is an isomorphism (b) f is not homomorphism

(c) f is onto (d) f is one to one

(6) If f be a homomorphism of group G onto G' with kernel K, then G' is :

(a) isomorphic to G/K (b) isomorphic to K/G

(c) isomorphic to G (d) isomorphic to G'/K

(7) If f(x) = (–1, –2, –3) and g(x) = (1, 3), then f(x) · g(x) is a polynomial of degree :

(a) 1 (b) 2

(c) 3 (d) 0

(8) An integral domain is :

(a) always a field (b) never a field

(c) a field when it is finite (d) None of these

(9) If $f(x) = (x - 2)^3 (x - 3)^2$, then 2 is the zero of polynomial f(x) of multiplicity :

(a) 2 (b) 3

(c) 5 (d) 1

(10) Which of the following polynomial is monic ?

(a) $(2x^2 + x + 1) \cdot (x^2 + 1)$ (b) $(2x^2 + x + 1) \cdot (\frac{1}{2} x^2 - x - 1)$

(c) $(2x^2 + x + 1) \cdot (- x - 1)$ (d) $(2x^2 + x + 1) \cdot (x^2 - 1)$

UNIT—I

2\. (a) Prove that subgroup H of group G is normal iff $H_a \cdot H_b = H_{ab}$; $\forall a, b \in G$. 4

(b) Show that if G is abelian group, then the quotient group G/N is also abelian group. Is its converse true, explain ? 3

(c) Let 'S_n' be a symmetric group of permutations of degree 'n' and 'A_n' be the set of all even permutation in 'S_n'. Then show that A_n is normal subgroup of 'S_n'. 3

3\. (d) If G = {1, –1, i, –i} and N = {1, –1}, then show that N is normal subgroup of multiplicative group G. Find quotient group G/N and also order of G/N. 4

(e) If H is subgroup of group G and N is normal subgroup of G, then show that $H \cap N$ is normal subgroup of H. 3

(f) Show that every subgroup of an abelian group is normal. 3

UNIT—II

4. (a) Let $\phi : G \rightarrow G'$ is an isomorphism. Then show that the order of an element $a \in G$ is equal to the order of its image $\phi(a) \in G'$. 4

(b) If $\phi : G \rightarrow G'$ is homorphism with Kernel K, then prove that K is normal subgroup of group G. 3

(c) Let G be any group and g be fixed element in G. Define $\phi : G \rightarrow G$ such that $\phi(x) = gxg^{-1}$, then prove that ϕ is an isomorphism of G onto G. 3

5. (d) If $\phi : G \rightarrow G'$ is a homomorphism with Kernel $K\phi$, then prove that ϕ is an isomorphism iff $K\phi = \{e\}$; where 'e' is an identity element of G. 4

(e) Let N be a normal subgroup of group G. Define a mapping $\phi : G \rightarrow G/N$ such that $\phi(x) = Nx$ for all $x \in G$. Then prove that ϕ is homomorphism of G onto G/N and Ker $\phi = N$. 3

(f) Let R be additive group of real numbers and R^+ be the multiplicative group of all positive real numbers. Then prove that the mapping $f : R \rightarrow R^+$ defined by $f(x) = e^x$; $\forall x \in R$ is an isomorphism. 3

UNIT—III

6. (a) Define commutative ring. If R is a ring with zero element '0', then for all a, b, c $\in$ R, prove that :

(i) $a \cdot 0 = 0 \cdot a = 0$

(ii) $a \cdot (-b) = (-a) \cdot b = -(ab)$

(iii) $a \cdot (b - c) = a \cdot b - a \cdot c$. 1+4

(b) Show that a ring is without zero divisor iff the cancellation laws holds in R. 5

7. (c) Show that the set 'S' of 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$; is a subring of ring of 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with respect to the operation addition and multiplication of the matrices; where a, b, c, d are the integers. 5

(d) Define Boolean ring. Prove that every Boolean ring is commutative. 1+4

UNIT—IV

8. (a) Prove that a nonempty subset K of a field F is a subfield of F iff

   (i) $a - b \in K$

   (ii) $a \cdot b^{-1} \in K$

   for all a, b $(\neq 0) \in K$. 5

   (b) Prove that the characteristic of an integral domain is either zero or a prime number. 5

9. (c) Prove that every finite integral domain is a field. 5

   (d) Prove that the characteristic of an integral domain is zero or $n > 0$ according as the order of any non-zero element regarded as member of the additive group of the integral domain is either infinity or n. 5

UNIT—V

10. (a) Prove that R is an integral domain iff R [x] is an integral domain. 4

   (b) If p / q is a rational zero of polynomial $f(x) = a_0 + a_1 x + \dots + a_n x^n$ and p and q have no common factor, then prove that p is a factor of a_0 and q is a factor of a_n. 4

   (c) Prove that the polynomial $f(x) = x^4 + 2x + 2$ is irreducible over the field of rational numbers. 2

11. (d) Let F be a field and p(x), q(x) be two non zero polynomials of F[x]. Then prove that

   $$\deg (p(x) \cdot q(x)) = \deg p(x) + \deg q(x).$$ 4

   (e) If p is a prime number, then prove that the polynomial $x^n - p$, is irreducible over the rationals. 3

   (f) If f(x) is a polynomial over a field F and $\alpha \in F$, then show that the remainder in the division of f(x) by $(x - \alpha)$ is $f(\alpha)$. 3