B.Sc. Part-II (Semester-IV) Examination

MATHEMATICS

(Modern Algebra : Groups and Rings)

Paper—VII

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt it once only.

(2) Solve **ONE** question from each unit.

1. Choose the correct alternatives (**1 mark each**) : 10

(i) Every transposition is an :

(a) Odd permutation (b) Even permutation

(c) Both odd and even (d) None of these

(ii) If G is a finite group and H is a subgroup of G, then :

(a) 0(H) + 0(G) (b) 0(H) – 0(G)

(c) 0(G)/0(H) (d) 0(H)/0(G)

(iii) Every cyclic group is :

(a) Abelian (b) Non-abelian

(c) Cyclic (d) Infinite cyclic

(iv) The order of the identity element e of any group G is :

(a) 0 (b) 1

(c) 2 (d) 3

(v) If f be a homomorphism of a group G onto G' with Kernel K, then G' is :

(a) isomorphic to G/K (b) isomorphic to K/G

(c) isomorphic to G (d) isomorphic to K

(vi) A homomorphism of a group G into itself is :

(a) Non-homomorphism (b) Isomorphism

(c) Endomorphism (d) None of these

(vii) The intersection of two subrings is a :

(a) Division ring (b) Subring

(c) Not subring (d) None of these

(viii) A finite integral domain is a :

(a) Field (b) Prime field

(c) Sub field (d) Proper field

(ix) The intersection of two left ideals of R is :

(a) A left ideal of R

(b) A right ideal of R

(c) Both left and right ideal of R

(d) None of these

(x) If U is an ideal of a ring R with unity 1 and $1 \in U$ then :

(a) $U = M$

(b) $U = R$

(c) $U \neq M$

(d) $U \neq R$

UNIT—I

2. (a) Show that if every element of the group G is its own inverse, then G is abelian. 4

(b) Show that the intersection of any two subgroups of a group G is a subgroup of G. 3

(c) Show that any two distinct cycles of a permutation of a finite set are disjoint. 3

3. (p) Prove that the system $(G, +)$ is an abelian group; with respect to '+'; where $G = \{x \mid x = a + b\sqrt{2}, a, b \in Q\}$. 4

(q) If G is a group, then prove that $(ab)^{-1} = b^{-1}a^{-1} \ \forall a, b \in G$. 3

(r) Prove that every cyclic group is abelian. 3

UNIT—II

4. (a) If G is a finite group and $a \in G$ then prove that $O(a) \mid O(G)$. 4

(b) Show that every subgroup of an abelian group is normal. 3

(c) Show that if G is abelian then the quotient group G/N is also abelian. Is its converse true ? 3

5. (p) A subgroup N of G is a normal subgroup of G iff the product of two right coset of N in G is again a right coset of N in G. Prove this. 4

(q) If N is a normal subgroup of G and H is any subgroup of G, then prove that NH is a subgroup of G. 3

(r) If $G = \{1, -1, i, -i\}$ and $N = \{1, -1\}$, then show that N is a normal subgroup of the multiplicative group G. Also find the quotient group G/N. 3

UNIT—III

6. (a) Prove that a homomorphism ϕ of G into G' with kernel K_ϕ is an isomorphism of G into G' if and only if $K_\phi = \{e\}$, where e is identity of G. 5

(b) If M, N are normal subgroups of G, then prove that :

$$\frac{NM}{M} \approx \frac{N}{N \cap M}.$$ 5

7. (p) If ϕ is a homomorphism of G into G' with kernel K, then prove that K is a normal subgroup of G. 5

(q) Let G be any group, g a fixed element in G. If $\phi : G \to G$ defined by $\phi(x) = gxg^{-1}$, then prove that ϕ is an isomorphism of G onto G. 5

UNIT—IV

8. (a) Define :

   (i) Integral domain

   (ii) Field.

   Prove that a field is an integral domain, but the converse is not true. 2+3

   (b) Prove that the characteristic of an integral domain is either zero or a prime number. 5

9. (p) Define commutative ring and prove that a ring R is commutative if and only if :

   $$(a + b)^2 = a^2 + 2ab + b^2.$$ 1+4

   (q) Prove that a non-empty subset S of a ring R is a subring of R if and only if $x - y, xy \in S \ \forall\ x, y \in S$. 5

UNIT—V

10. (a) If R be a ring with unit elements and R not necessarily commutative such that the only right ideals of R are {0} and R, then prove that R is a division ring. 5

    (b) If U is an ideal of a ring R, then prove that R/U is a homomorphic image of R. 5

11. (p) If U and V are ideals of a ring R, then prove that :

    (i) $U \cap V$ is an ideal of R

    (ii) $U \cap V$ is the largest ideal that is contained in both U and V. 5

    (q) Let $U = \{19n \mid n \in Z\}$ be an ideal of the ring of integers Z and V be an ideal of Z with $U \subset V \subset Z$. Prove that V = U or V = Z i.e. U is a maximal ideal of Z. 5