AU–104

B.Sc. (Part-II) Semester-IV Examination

MATHEMATICS

(Modern Algebra Groups and Rings)

Paper—VII

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt at once only.

(2) Solve **ONE** question from each unit.

1. Choose the correct alternatives (**1** mark each) : 10

(i) A nonempty subset H of the group G is a subgroup of G if and only if a, b $\in$ H $\Rightarrow$

(a) $(ab)^{-1} \in H$ (b) $ab^{-1} \in H$

(c) $a^{-1}b^{-1} \in H$ (d) None of these

(ii) The product of two even permutation is :

(a) Odd (b) Even

(c) Both odd and even (d) None of these

(iii) If G is a finite group and N is a normal subgroup of G, then O(G/N) is equal to :

(a) $O(G) \cdot O(N)$ (b) $O(G) + O(N)$

(c) $O(G) / O(N)$ (d) $O(G) - O(N)$

(iv) The subgroup N of G is a normal subgroup of G iff :

(a) $gN \neq Ng$ for some $g \in G$ (b) $gN = Ng$ for all $g \in G$

(c) $Ng = N$ for some $g \in G$ (d) $gN = N$ for all $g \in G$

(v) Let (G, +) be a group. Then mapping $\phi : G \to G$ is homomorphism if :

(a) $\phi(a + b) = \phi(a) + \phi(b)$ (b) $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$

(c) $\phi(a - b) = \phi(a) - \phi(b)$ (d) $\phi\left(\frac{a}{b}\right) = \phi(a)/\phi(b)$

(vi) If ϕ be a homomorphism of group G onto G' with Kernel K, then G' is :

(a) Isomorphic to G/K (b) Isomorphic to K/G

(c) Isomorphic to G (d) One-one homomorphism

(vii) A division ring must contain at least :

(a) One element (b) Two elements

(c) Three elements (d) None of these

(viii) If in a ring R, $x^2 = x \ \forall x \in R$, then R is :

(a) Commutative ring (b) Division ring

(c) Boolean ring (d) Ring with unity

(ix) If U is an ideal of a ring R with unity 1 and $1 \in U$ then :

(a) $U = R$ (b) $U \neq R$

(c) $U = M$ (d) None of these

(x) A ring R has maximal ideals :

(a) If R is finite

(b) If R is finite with at least 2 elements

(c) Only if R is finite

(d) None of these

UNIT—I

2. (a) If G is an abelian group, then prove that :

$(ab)^n = a^n b^n \ \forall a, b \in G$ and $\forall$ integers n. 5

(b) Prove that intersection of any two subgroups of group is also a subgroup. 3

(c) If G is a group, then prove that for every $a \in G$, $(a^{-1})^{-1} = a$. 2

3. (p) If G is a group in which $(ab)^i = a^i b^i$ for three consecutive integers i for all $a, b \in G$, then prove that G is abelian. 4

(q) Prove that every permutation is a product of 2-cycles or transpositions. 4

(r) Prove that the identity of a group G is unique. 2

UNIT—II

4. (a) Prove that the subgroup N of G is a normal subgroup of G if and only if each left coset of N in G is a right coset of N in G. 4

(b) Let H be a subgroup of G. If $N(H) = \{g \in G / gHg^{-1} = H\}$ then prove that N(H) is a subgroup of G. 4

(c) Show that if G is abelian, then the quotient group G/N is also abelian. 2

5. (p) Let H be a subgroup of a group G. Let for $g \in G$,

$$gHg^{-1} = \{ghg^{-1} / h \in H\}$$

prove that gHg^{-1} is a subgroup of G. 4

(q) If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G. 3

(r) If H is a subgroup of G and N is a normal subgroup of G then prove that $H \cap N$ is a normal subgroup of H. 3

UNIT—III

6. (a) If ϕ is a homomorphism of a group G into a group G', then prove that :

(i) $\phi(e) = e'$

(ii) $\phi(x^{-1}) = (\phi(x))^{-1} \ \forall \ x \in G$

where e and e' are the unit elements of G and G' respectively. 4

(b) Prove that a homomorphism ϕ of G into G' with Kernel K_ϕ is an isomorphism of G into G' if and only if $K_\phi = \{e\}$, where e = identity of G. 3

(c) Let N be a normal subgroup of G. Define the mapping $\phi : G \to G/N$ such that $\phi(x) = Nx, \ \forall \ x \in G$. Then prove that ϕ is a homomorphism of G onto G/N. 3

7. (p) If ϕ be a homomorphism of G onto G' with Kernel K. Then prove that $G/K \approx G'$. 5

(q) Let ϕ be a homomorphism of G onto G' with Kernel K. Let N' be a normal subgroup of G' and $N = \{x \in G / \phi(x) \in N'\}$. Then prove that $\frac{G}{N} \approx \frac{G'}{N'}$. 5

UNIT—IV

8. (a) Prove that the set of units in a commutative ring with unity is a multiplicative abelian group. 4

(b) Let K be a nonempty subset of a field F. Then prove that K is a subfield of F if and only if $x - y, xy^{-1} \in K \ \forall \ x, y \in K, y \neq 0$. 4

(c) Define :

(i) Prime field

(ii) Ring with no zero divisor. 1+1

9. (p) Let R be a ring with a unit element 1, in which $(ab)^2 = a^2b^2 \ \forall \ a, b \in R$. Prove that R must be commutative. 5

(q) If R is a ring in which $x^2 = x \ \forall \ x \in R$, then prove that R is a commutative ring of characteristic 2. 3+2

UNIT—V

10. (a) If U is an ideal of the ring R, then prove that R/U is a ring. 4

(b) Prove that a homomorphism f of a ring R to a ring R' is an isomorphism iff Ker f = {0}. 4

(c) Define :

(i) Trivial Ideals

(ii) Simple Ring. 1+1

11. (p) If F is a field, then prove that its only ideals are {0} and F itself. 3

(q) Let R be a commutative ring and P an ideal of R. Prove that the ring of residue classes R/P is an integral domain iff P is a prime ideal. 5

(r) If U is a left ideal of a ring R, then prove that U is a subring of R. 2