AV–1759

B.Sc. (Part—II) Semester-IV Examination
MATHEMATICS
Paper–VII
(Modern Algebra Groups and Rings)

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt it once only.

(2) Solve **ONE** question from each unit.

1. Choose the correct alternative (1 mark each) : 10

(i) A group having only improper normal subgroup is called _________.

(a) a finite group (b) a permutation group

(c) a simple group (d) None of these

(ii) Every subgroup of a cyclic group is _________.

(a) non abelian (b) cyclic

(c) cyclic but not abelian (d) abelian but not cyclic

(iii) The identity permutation is _________.

(a) even (b) odd

(c) even and odd (d) even or odd

(iv) Let G be a group. Then $(ab)^{-1}$ is equal to _________.

(a) $a^{-1}b^{-1}$ (b) $b^{-1}a^{-1}$

(c) $(ba)^{-1}$ (d) None of these

(v) A homomorphism of a group into itself is _________.

(a) a homomorphism (b) an isomorphism

(c) an endomorphism (d) None of these

(vi) An integral domain has at least _________.

(a) One element (b) Two element

(c) Three element (d) None of these

(vii) If in a ring R, $x^2 = x \ \forall \ x \in R$, then R is _________.

(a) Commutative ring (b) Division ring

(c) Boolean ring (d) Ring with unity

(viii) A field which contains no proper subfield is called _________.

(a) Sub field (b) Prime field

(c) Integral domain (d) Division ring

(ix) The intersection of two left ideals of a ring R is _________.

(a) left ideal of R (b) right ideal of R

(c) both (a) and (b) (d) None of these

(x) The characteristic of an integral domain is :

(a) even number (b) odd number

(c) prime number (d) None of these.

UNIT–I

2. (a) Prove that the set $G = \{1, W, W^2\}$ is a group w.r.t. multiplication. 4

(b) Prove that every subgroup of a cyclic group is cyclic. 4

(c) If $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ then prove that $f.g \neq g.f$. 2

3. (p) Let G be a group. Prove that a non-empty subset H of G is a subgroup of G iff $a, b \in H \Rightarrow a.b^{-1} \in H$. 4

(q) Find whether the following permutations are even or odd : 4

(i) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$

(ii) $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix}$

(r) Define : 2

(i) Cyclic group

(ii) Order of an element of a group.

UNIT–II

4. (a) If H is a subgroup of a group G, then prove that any two right (left) cosets of H in G are either identical or disjoint. 5

(b) Prove that N is a normal subgroup of G if and only if $gNg^{-1} = N \; \forall \; g \in G$. 5

5. (p) Show that if G is abelian, then the quotient group G/N is also abelian. 3

(q) Let H be a subgroup of G and $N(H) = \{g \in G \mid gHg^{-1} = H\}$. Show that H is normal in G iff $N(H) = G$. 4

(r) Prove that the intersection of two normal subgroups of a group is a normal subgroup of G. 3

UNIT–III

6. (a) If ϕ is a homomorphism of G into G′ with Kernel K, then prove that K is a normal subgroup of G. 4

(b) If ϕ is homomorphism of a group G into a group G′, then prove that :

(i) $\phi(e) = e'$ and

(ii) $\phi(x^{-1}) = (\phi(x))^{-1} \; \forall \; x \in G$

where e and e′ are identities of G and G′ respectively. 3

(c) Let G be a group of real numbers under addition and $\phi : G \rightarrow G$ such that $\phi(x) = 13x \; \forall \; x \in G$, then prove that ϕ is homomorphism. 3

7. (p) If ϕ is homomorphism of G onto G′ with Kernel K, then prove that $G/K \approx G'$. 5

(q) Define :

(i) Homomorphism

(ii) Kernel of homomorphism.

Prove that any Kernel is non-empty. 2+3

UNIT–IV

8. (a) Prove that the intersection of any family of subrings of a ring R is a sub ring of R. 3

   (b) If in a ring R, $x^3 = x \; \forall \; x \in R$, then show that R is commutative. 4

   (c) Let the characteristic of the ring R be 2 and let $ab = ba \; \forall \; a, b \in R$ then show that $(a + b)^2 = a^2 + b^2$. 3

9. (p) Prove that Prime field of characteristic zero is isomorphic to the field Q of rational numbers. 5

   (q) Let R be a ring with a unit element, 1, in which $(ab)^2 = a^2b^2 \; \forall \; a, b \in R$. Then prove that R is commutative. 5

UNIT–V

10. (a) If U is an ideal of a ring R with unity 1 and $1 \in U$, then prove that U = R. 2

    (b) If R is a commutative ring with a unit element and M is an ideal of R, then prove that M is a Maximal ideal of R iff R|M is a field. 5

    (c) Let R be a commutative ring with unity. Then prove that every maximal ideal of R is a prime ideal. 3

11. (p) If U is an ideal of ring R, then prove that R|U is a homomorphic image of R. 4

    (q) Let M be the ring of matrices of order 2 over the field R of real numbers and $U = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in R \right\}$. Prove that U is a right ideal of M but U is not left ideal. 3

    (r) Let $U = \{19n \mid n \in z\}$ be an ideal of the ring of integers Z and V be an ideal of Z with $U \subset V \subset Z$. Then prove that V = U or V = Z. 3