AV–1733

B.Sc. (Part—II) Semester—III Examination

MATHEMATICS

(Elementary Number Theory)

Paper—VI

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory, attempt it once only.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative (**1** mark each) :

(i) The number of multiples of 10^{44} that divides 10^{55} is :

(a) 144 (b) 11

(c) 121 (d) 12

(ii) The integers of the form $2^{2^n}+1$ are called the :

(a) Prime number (b) Ramanuj number

(c) Fermat number (d) Real number

(iii) If p is prime then $2^p + 3^p$ is :

(a) Perfect square (b) Not perfect square

(c) Positive integer (d) Negative integer

(iv) A solution of $ax \equiv 1 \pmod{m}$ with $(a, m) = 1$ is called an :

(a) Even number (b) Odd number

(c) Modulo m (d) Inverse of modulo m

(v) What is the total number of solutions in integers to the equation $3x + 5y = 21$?

(a) 0 (b) ∞

(c) 2 (d) 4

(vi) The set {0, 1, 2, 3} is complete system of residues modulo :

(a) 3 (b) 4

(c) 5 (d) 2

(vii) The function f is multiplicative if :

(a) $f(m \cdot n) = f(m)|f(n)$ (b) $f(mn) = f(n)|f(m)$

(c) $f(mn) = f(m)\,f(n)$ (d) None of these

(viii) The number of primitive roots of 53 is :

(a) Zero (b) One

(c) Twenty (d) Twenty Four

(ix) The equation $m^2 - 33n + 1 = 0$, where m, n are integers, has :

(a) Exactly one solution (b) No solution

(c) Infinitely many solutions (d) Exactly two solutions

(x) The number of solutions to the congruence $x^3 \equiv 3 \pmod 7$ is :

(a) 3 (b) 2

(c) 1 (d) No solution 10

UNIT—I

2. (a) Let a and b be integers that both are not zero. Then prove that a and b are relatively prime iff there exist integers x and y such that $xa + yb = 1$. 4

(b) If $(a, b) = 1$, show that $(a + b, a - b) = 1$ **or** 2. 3

(c) If $a|bc$ and $(a, b) = 1$, then prove that $a|c$. 3

3. (p) State and prove Division Algorithm Theorem. 1+3

(q) If x and y are odd then prove that $x^2 + y^2$ is not a perfect square. 3

(r) Show that the square of every odd integer is of the form $8m + 1$. 3

UNIT—II

4. (a) Prove that there are infinite number of primes. 5

(b) Find the gcd and lcm of a = 18,900 and b = 17,160 by writing each of the numbers a and b in prime factorization canonical form. 5

5. (p) Prove that for all positive integer n,

$$F_0 F_1 \ldots F_{n-1} = F_n - 2.$$ 5

(q) Let a and b be relatively prime integers. If d is a positive divisor of ab, then show that there is a unique pair of positive divisors d_1 of a and d_2 of b such that $d = d_1 d_2$. 5

UNIT—III

6. (a) Solve the congruence :

$$7x \equiv 3 \pmod{12}.$$ 4

(b) If $a \equiv b \pmod{m_1}$ and $a \equiv b \pmod{m_2}$, then prove that $a \equiv b \pmod{[m_1, m_2]}$. 4

(c) If a, b, c are integers such that $a \equiv b \pmod m$, then prove that $(a + c) \equiv (b + c) \pmod m$. 2

7. (p) If $r_1, r_2, \ldots, r_m$ is a complete system of residues modulo m and $(a, m) = 1$, a is a positive integer, then prove that $ar_1 + b, ar_2 + b, \ldots, ar_m + b$ is also complete system of residues modulo m. 5

(q) If f is a polynomial with integral coefficients and $a \equiv b \pmod m$, then prove that :

$$f(a) \equiv f(b) \pmod m.$$ 5

UNIT—IV

8. (a) Define multiplicative function. If f is a multiplicative function and $n = p_1^{a_1} \cdot p_2^{a_2} \ldots . p_m^{a_m}$ is the prime-power factorization of the positive integer n, then prove that $f(n) = f(p_1^{a_1}) \cdot f(p_2^{a_2}) \ldots . f(p_m^{a_m})$. 1+4

(b) If n is a positive integer, then prove that $\sum_{d|n} \phi(d) = n$. 5

9. (p) Prove that for each positive integer $n \geq 1$, $\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$. 5

(q) Solve the linear congruences using Euler's theorem $5x \equiv 3 \pmod{14}$. 5

UNIT—V

10. (a) Let p be an odd prime and let a be an integer with (a, p) = 1 then prove that $(a/p) \equiv a^{(p-1)/2} \bmod (p)$. 5

(b) Solve the quadratic congruence $x^2 + 7x + 10 \equiv 0 \pmod{11}$. 5

11. (p) Prove that if p is an odd prime, then $x^2 \equiv 2 \pmod{p}$ has solution iff $p \equiv \pm 1 \pmod{8}$. 5

(q) Find all primitive roots of p = 41. 5